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Real-time simulation of biological soft tissues: a PGD approach

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SUMMARY

We introduce here a novel approach for the numerical simulation of non-linear, hyperelastic soft tissues at kHz feedback rates, necessary for haptic rendering. This approach is based upon the use of proper generalized decomposition (PGD) techniques, a generalization of proper orthogonal decompositions (POD). PGD techniques can be considered as a means of *a priori* model order reduction and provides with a physics-based meta-model without the need for prior computer experiments. The suggested strategy is thus composed by a off-line phase, in which a general meta-model is computed, and an on-line evaluation phase in which the results are obtained at real time. Results are provided that show the potential of the proposed technique, together with some benchmark test that show the accuracy of the method. Copyright © 2012 John Wiley & Sons, Ltd.

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KEY WORDS: Real time, surgery, model order reduction, proper generalized decomposition.

CONTENTS

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1. INTRODUCTION

Real-time simulation is one of the most challenging scenarios for simulation-based engineering sciences (SBES). The term *real time* strongly depends on the particular pursued application, but surgery simulation is among the most restrictive ones. Haptic surgery simulators compute the response of biological soft tissues and give it back to the peripherals at, at least, 25 Hz of feedback rate if visual realism is needed, and, notably, 500Hz-1kHz if haptic (force) response is desired [18] [17]. But biological soft tissues are known to be highly non-linear [19] [2] [23], very frequently modeled in a hyperelastic framework. It is well-known, in addition, that at least non-linear strain measures should be incorporated into the simulation, otherwise performing arbitrarily bad in terms of visual perception, spurious gain in volume, etc [17].

This feedback rate for non-linear problems constitutes indeed a great challenge for nowadays simulation techniques, based upon finite element methods. This is perhaps the ultimate reason for the lack, up to our knowledge, of surgery simulators of the second generation [18], i.e., those that incorporate state-of-the-art constitutive modelling of soft tissues to the simulation. A few references can be cited that incorporate non-linear tissue behaviour (mainly Kirchhoff-Saint Venant hyperelasticity), see [33] [32] [26]. Many of them are indeed based upon explicit finite elements, possibly implemented in graphics processing units (GPU). This is due to the very astringent conditions dictated by the haptic feedback rates, that prevent the use, in nowadays computers, of standard Newton-like methods for the solution of non-linear systems of equations.

Recently, model order reduction has been seen as a powerful means to achieve real-time performance in the simulation of non-linear solids. For instance, Barbič and James [7] proposed recently a method for model order reduction of Kirchhoff-Saint Venant solids at haptic feedback rates. For more sophisticated constitutive laws, recent work on Proper Orthogonal Decomposition methods showed that visual feedback performance can be easily obtained, but this is not the case very often if we need haptic feedback [27] [29] [34]. In this case, linearization schemes for the resulting non-linear systems of equations is mandatory, although not straightforward to accomplish. Very often, Newton strategies are out of reach, due to the inherent lack of time employed during stiffness matrix updates. In [30] [28] a different strategy was developed that employed a combination of POD techniques with Asymptotic Numerical Methods [15] [1] [12] [16]. In that case, no tangent stiffness matrix updates are necessary, and only a set of linear problems, all with the same stiffness matrix, must be solved.

Proper Generalized Decomposition (PGD) methods, on the contrary, arose recently as a means to overcome the so-called *curse of dimensionality* associated to problems defined in high dimensional spaces [6] [5] [13] [14] [25]. Although the origins of the technique can be traced back to the LArge Time INcrement method, LATIN method and its associated radial loading [24], PGD is nowadays seen as a powerful method of model order reduction that generalizes POD (and hence its name). Its field of application has gone far beyond the initial objectives, and it is now being applied in a variety of fields: simulation-based control of processes [20] [21], efficient simulation of plates and shells [8], simulation of gene regulatory networks [4], to cite a few.

The main idea for the extension of a technique initially developed to deal with high dimensional problems to a more general setting lies precisely in its ability to treat standard models as if they were multidimensional. The key idea was initially put forth in [31]. In it, parametric equations were cast into a multidimensional setting, thus taking advantage of the PGD solution structure. Going one step forward, PGD can advantageously be employed into real-time simulation frameworks by simply considering all parameters, but also all possible boundary conditions (including initial boundary conditions, see [22]) as new dimensions of the problem.

The resulting PGD solution to the problem is expressed as a finite sum of separable functions that provides actually a meta-model for the problem, for which no prior computer experiment (also known as *snapshots* in the model order reduction community) is necessary. This meta-model can then be successfully applied in real time to obtain the response of the system at kHz rates, as will be demonstrated in subsequent sections of this paper. This approach allows even to solve models on

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handheld devices such as tablets and smartphones, thus opening the range of possible applications of the technique.

In this paper we present a novel technique for the simulation of biological soft tissues under hyperelasticity assumptions at haptic feedback rates. It is based on the use of the before mentioned PGD approach and an explicit linearization of the weak form generated by non-linear strain measures. The use of PGD therefore allows not only for a complete generalization of previous works in the field (see [27] [29]) but also allows to a completely new formulation of the problem. For the problem at hand, POD techniques compute the solution (the so-called snapshots) of different complete models for different contact positions between surgical tool —scalpel– and organ. By performing a statistical analysis on top of these results, POD techniques extract the so-called *m*odes, i.e., those displacement fields that best represent the solution of the complete problems. Theses modes, took as Ritz-like, global basis, are then used to approximate problems different to the original ones, i.e., for contact positions not initially considered. Therefore, the design of an appropriate simulation, or computer experiment, campaign is the key aspect of the method. Also, how to efficiently and accurately interpolate among reduced basis is another crucial aspect, not fully resolved.

On the contrary, PGD methods consider a parametric problem. In this case, the parameter is the contact position. By formulating the parametric problem as a high-dimensional one, PGD methods allows for the efficient solution and ulterior storage of the model in the form of a sum of separable functions. Therefore, rather than creating reduced models for particular positions of the tool, PGD computes a general solution for *a*ny position of the tools, so that no subsequent interpolation of reduced models is necessary, nor the computation of snapshots. Thus, the proposed method is based upon an off-line phase in which this general solution is computed, and an on-line one in which the solution is only evaluated at impressive feedback rates, here on the order of kHz.

The paper is organized as follows. In Section 2 we introduce the basics of Proper Generalized Decomposition applied to the problem of a hyperelastic solid under moving punctual loads, which the most frequent case in surgery simulation. In Section 3 a very simple linearization of the non-linear problem is introduced that allows for a simple yet effective computation of the PGD approach to the problem. Although this simple linearization is by no means the only possible one, its performance is analyzed in Section 4, through a series of benchmark problems. It is shown how the PGD approach to the problem of real-time simulation of soft tissue deformation opens new insights on how the problem can be attacked.

2. A PGD APPROACH TO VIRTUAL SURGERY

As already mentioned in the introduction, the key issue in the usage of PGD approaches for real time simulation, and the one that makes it completely different in spirit from POD, lies in the formulation of the original problem as a parametric one. This parametric problem is then re-formulated as a high dimensional problem by considering each parameter as a new coordinate in the state space. The PGD method then looks for an effective solution in the form of a finite sum of separable functions, so as to be able to avoid the curse of dimensionality associated to high dimensional problems and mesh-based discretization techniques.

In this framework, the problem of determining the response of an organ to the load transmitted by the contact with a surgical tool could be formulated as to determine the displacement at any point of the model, $\boldsymbol{u}(x, y, z)$, for any load position $\boldsymbol{s}$ and for any force vector orientation and module, $\boldsymbol{t}$, thus rendering a problem defined in the physical space ($\mathbb{R}^3$), plus a six-dimensional state space ($\mathbb{R}^6$).

For the sake of simplicity in the following development, and without loss of generality, we assume a load vector $\boldsymbol{t}$ with unit module and oriented in the vertical direction. This renders a problem defined in $\mathbb{R}^6$ ($\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{s})$), with all the characteristics of the before-mentioned one.

Let us consider the weak form of the equilibrium equations (balance of linear momentum). Again, for the sake of simplicity, we omit inertia terms. The interested reader could consult [22] for the treatment of Ordinary Differential Equations (ODEs) in the framework of PGD. Under these

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assumptions, the weak form of the problem, extended to the whole geometry of the organ, Ω and the portion of its boundary which is accessible to the surgeon, $\bar{\Gamma} \subset \Gamma_t$[†], consists in finding the displacement $\boldsymbol{u} \in \mathcal{H}^1$ such that for all $\boldsymbol{u}^* \in \mathcal{H}_0^1$:

$$\int_{\bar{\Gamma}} \int_{\Omega} \boldsymbol{\nabla}_s \boldsymbol{u}^* : \boldsymbol{\sigma} d\Omega d\bar{\Gamma} = \int_{\bar{\Gamma}} \int_{\Gamma_{t2}} \boldsymbol{u}^* \cdot \boldsymbol{t} d\Gamma d\bar{\Gamma} \tag{1}$$

where $\Gamma = \Gamma_u \cup \Gamma_t$ represents the boundary of the organ, divided into essential and natural regions, and where $\Gamma_t = \Gamma_{t1} \cup \Gamma_{t2}$, i.e., regions of homogeneous and non-homogeneous, respectively, natural boundary conditions. Here, $\boldsymbol{t} = \boldsymbol{e}_k \cdot \delta(\boldsymbol{x} - \boldsymbol{s})$, where δ represents the Dirac-delta function and $\boldsymbol{e}_k$ the unit vector along the z-coordinate axis (we consider here, for the ease of exposition, a unit load directed towards the negative z axis of reference).

Once regularized, the Dirac-delta term is approximated by a truncated series of separable functions in the spirit of the PGD method, i.e.,

$$t_j \approx \sum_{i=1}^{m} f_j^i(\boldsymbol{x}) g_j^i(\boldsymbol{s}) \tag{2}$$

where m represents the order of truncation and f_j^i, g_j^i represent the j-th component of vectorial functions in space and boundary position, respectively.

The PGD approach to the problem is characterized by the construction, in an iterative way, of an approximation to the solution in the form of a finite sum of separable functions. Assume that we have converged to a solution, at iteration n of this procedure,

$$u_j^n(\boldsymbol{x}, \boldsymbol{s}) = \sum_{k=1}^{n} X_j^k(\boldsymbol{x}) \cdot Y_j^k(\boldsymbol{s}), \tag{3}$$

where the term u_j refers to the j-th component of the displacement vector, $j = 1, 2, 3$ and functions $\boldsymbol{X}^k$ and $\boldsymbol{Y}^k$ represent the separated functions used to approximate the unknown field, obtained in previous iterations of the PGD algorithm.

If we look for an improvement of this approximation, the $(n+1)$-th term will look like

$$u_j^{n+1}(\boldsymbol{x}, \boldsymbol{s}) = u_j^n(\boldsymbol{x}, \boldsymbol{s}) + R_j(\boldsymbol{x}) \cdot S_j(\boldsymbol{s}), \tag{4}$$

where $\boldsymbol{R}(\boldsymbol{x})$ and $\boldsymbol{S}(\boldsymbol{s})$ are the sought functions that improve the approximation.

In this framework, the admissible variation of the displacement will be given by

$$u_j^*(\boldsymbol{x}, \boldsymbol{s}) = R_j^*(\boldsymbol{x}) \cdot S_j(\boldsymbol{s}) + R_j(\boldsymbol{x}) \cdot S_j^*(\boldsymbol{s}). \tag{5}$$

At this point several options are at hand so as to determine the new pair of functions $\boldsymbol{R}$ and $\boldsymbol{S}$. The most frequently used, due to both its easy of implementation and good convergence properties, in general, is a fixed-point algorithm in which functions $\boldsymbol{R}$ and $\boldsymbol{S}$ are sought iteratively. We describe briefly the implementation of this algorithm.

2.1. Computation of $S(\boldsymbol{s})$ assuming $R(\boldsymbol{x})$ is known

In this case, following standard assumptions of variational calculus, we have

$$u_j^*(\boldsymbol{x}, \boldsymbol{s}) = R_j(\boldsymbol{x}) \cdot S_j^*(\boldsymbol{s}), \tag{6}$$

[†]Typically, of all of the natural region of the boundary, $\Gamma_t \approx \Gamma$ —which in this case coincides with virtually all the boundary of organs, $\Gamma = \partial\Omega$, since they are not fixed or clamped, but in contact to other organs— only a portion are accessible to the surgeon. In minimally invasive surgery, surgeons operate through a small incision on the skin of the patient, having access only to a limited portion of the boundary, here termed $\bar{\Gamma}$.

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or, equivalently, $\boldsymbol{u}^*(\boldsymbol{x},\boldsymbol{s}) = \boldsymbol{R} \circ \boldsymbol{S}^*$, where the symbol "$\circ$" denotes the so-called entry-wise, Hadamard or Schur multiplication for vectors. Once substituted into Eq. (1), gives

$$\int_{\bar{\Gamma}} \int_{\Omega} \boldsymbol{\nabla}_s(\boldsymbol{R} \circ \boldsymbol{S}^*) : \mathbf{C} : \boldsymbol{\nabla}_s \left(\sum_{k=1}^{n} \boldsymbol{X}^k \circ \boldsymbol{Y}^k + \boldsymbol{R} \circ \boldsymbol{S} \right) d\Omega d\bar{\Gamma} = \int_{\bar{\Gamma}} \int_{\Gamma_{t2}} (\boldsymbol{R} \circ \boldsymbol{S}^*) \cdot \left(\sum_{k=1}^{m} \boldsymbol{f}^k \circ \boldsymbol{g}^k \right) d\Gamma d\bar{\Gamma}, \quad (7)$$

or, equivalently (we omit obvious functional dependencies)

$$\begin{aligned} &\int_{\bar{\Gamma}} \int_{\Omega} \boldsymbol{\nabla}_s(\boldsymbol{R} \circ \boldsymbol{S}^*) : \mathbf{C} : \boldsymbol{\nabla}_s(\boldsymbol{R} \circ \boldsymbol{S}) d\Omega d\bar{\Gamma} \\ &= \int_{\bar{\Gamma}} \int_{\Gamma_{t2}} (\boldsymbol{R} \circ \boldsymbol{S}^*) \cdot \left(\sum_{k=1}^{m} \boldsymbol{f}^k \circ \boldsymbol{g}^k \right) d\Gamma d\bar{\Gamma} - \int_{\bar{\Gamma}} \int_{\Omega} \boldsymbol{\nabla}_s (\boldsymbol{R} \circ \boldsymbol{S}^*) \cdot \mathcal{R}^n d\Omega d\bar{\Gamma}, \end{aligned} \quad (8)$$

where $\mathcal{R}^n$ represents:

$$\mathcal{R}^n = \mathbf{C} : \boldsymbol{\nabla}_s \boldsymbol{u}^n. \quad (9)$$

Since the symmetric gradient operates on spatial variables only, we have:

$$\begin{aligned} &\int_{\bar{\Gamma}} \int_{\Omega} (\boldsymbol{\nabla}_s \boldsymbol{R} \circ \boldsymbol{S}^*) : \mathbf{C} : (\boldsymbol{\nabla}_s \boldsymbol{R} \circ \boldsymbol{S}) d\Omega d\bar{\Gamma} \\ &= \int_{\bar{\Gamma}} \int_{\Gamma_{t2}} (\boldsymbol{R} \circ \boldsymbol{S}^*) \cdot \left(\sum_{k=1}^{m} \boldsymbol{f}^k \circ \boldsymbol{g}^k \right) d\Gamma d\bar{\Gamma} - \int_{\bar{\Gamma}} \int_{\Omega} (\boldsymbol{\nabla}_s \boldsymbol{R} \circ \boldsymbol{S}^*) \cdot \mathcal{R}^n d\Omega d\bar{\Gamma} \end{aligned} \quad (10)$$

where all the terms depending on $\boldsymbol{x}$ are known and hence we can compute all integrals over Ω and Γ_{t2} (support of the regularization of the initially punctual load) to derive an equation to compute $\boldsymbol{S}(\boldsymbol{s})$.

2.2. Computation of $R(\boldsymbol{x})$ assuming $S(\boldsymbol{s})$ is known

Equivalently, in this case, we have

$$u_j^*(\boldsymbol{x},\boldsymbol{s}) = R_j^*(\boldsymbol{x}) \cdot S_j(\boldsymbol{s}), \quad (11)$$

which, once substituted into Eq. (1), gives

$$\int_{\bar{\Gamma}} \int_{\Omega} \boldsymbol{\nabla}_s(\boldsymbol{R}^* \circ \boldsymbol{S}) : \mathbf{C} : \boldsymbol{\nabla}_s \left(\sum_{k=1}^{n} \boldsymbol{X}^k \circ \boldsymbol{Y}^k + \boldsymbol{R} \circ \boldsymbol{S} \right) d\Omega d\bar{\Gamma} = \int_{\bar{\Gamma}} \int_{\Gamma_{t2}} (\boldsymbol{R}^* \circ \boldsymbol{S}) \cdot \left(\sum_{k=1}^{m} \boldsymbol{f}^k \circ \boldsymbol{g}^k \right) d\Gamma d\bar{\Gamma}. \quad (12)$$

In this case all the terms depending on $\boldsymbol{s}$ (load position) can be integrated over $\bar{\Gamma}$, leading to a generalized elastic problem to compute function $\boldsymbol{R}(\boldsymbol{x})$.

This simple algorithm renders, in general, excellent convergence properties (see [14] and references therein).

3. ONE POSSIBLE EXPLICIT LINEARIZATION OF THE FORMULATION

Formulation introduced in Section 2 assumes implicitly small strains. But this assumption has been found to be very insufficient for virtual surgery. Strains are large very often, and when solved in a

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small strain setting, organs appear to suffer an unphysical gain in volume that renders the simulations clearly non-realistic [18]. Although real-time simulation of surgery does not look nowadays for accuracy levels similar to those common in usual engineering practice (quoting Cotin and Bro-Nielsen, [11], "... the model may be physically correct if it looks right") the inclusion of non-linear strain measures seems to be crucial.

Soft tissues are frequently formulated under hyperelasticity assumptions [23]. Again, for the sake of simplicity, we refer ourselves to a Kirchhoff-Saint Venant constitutive framework. Despite being very limited (and even unstable under compression due to the lack of polyconvexity of the strain energy functional), Kirchhoff-Saint Venant constitutive equations are widely used at this moment for real-time simulation of soft tissues, see [35] [30] [32], among others.

The Kirchhoff-Saint Venant model is characterized by the energy density functional given by

$$\Psi = \frac{\lambda}{2}(\text{tr}(\boldsymbol{E}))^2 + \mu \boldsymbol{E} : \boldsymbol{E} \tag{13}$$

where λ and μ are Lame's constants. The Green-Lagrange strain tensor, $\boldsymbol{E}$, has the form

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - 1) = \boldsymbol{\nabla}_s\boldsymbol{u} + \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}^T) \tag{14}$$

where $\boldsymbol{F} = \boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{I}$ is the gradient of deformation tensor. The second Piola-Kirchhoff stress tensor can be obtained by

$$\boldsymbol{S} = \frac{\partial\Psi(\boldsymbol{E})}{\partial\boldsymbol{E}} = \mathbf{C} : \boldsymbol{E} \tag{15}$$

in which $\mathbf{C}$ is the fourth-order constitutive (elastic) tensor.

Very little has been written about PGD approximations for non linear solid mechanics problems, other that Ladeveze's works in complex thermomechanical non-linear models [24]. Here we focus in the non-linear and parametric case within a fully separated representation. In this case, we restrict ourselves to quasi-static problems, for the sake of simplicity, and therefore introduce a pseudo-time $t \in [0, 1]$ to perform the linearization.

Consistent linearizations of the resulting set of equations in the framework of PGD approximations are far from being trivial, so here we keep the formulation as simple as possible by performing a simple explicit linearization of the weak form of the problem.

Thus, load is applied along a series of time increments Δt, provoking increments in the displacement $\Delta\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{s})$. At each time increment, a PGD fixed point alternating directions algorithm similar to those introduced in Section 2 is employed. So, if we introduce the non-linear strain measure given by Eq. (14), into this incremental framework, we have (we omit obvious functional dependencies for clarity)

$$\boldsymbol{E}^{t+\Delta t} = \boldsymbol{\nabla}_s\left(\boldsymbol{u}^t + \Delta\boldsymbol{u}\right) + \frac{1}{2}\left(\boldsymbol{\nabla}(\boldsymbol{u}^t + \Delta\boldsymbol{u})\cdot\boldsymbol{\nabla}^T(\boldsymbol{u}^t + \Delta\boldsymbol{u})\right). \tag{16}$$

Similarly, admissible variation of strain reads

$$\begin{aligned}\boldsymbol{E}^* &= \boldsymbol{\nabla}_s(\Delta\boldsymbol{u}^*) + \frac{1}{2}(\boldsymbol{\nabla}(\Delta\boldsymbol{u}^*))\cdot\boldsymbol{\nabla}^T(\boldsymbol{u}^t + \Delta\boldsymbol{u}) + \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}^t + \Delta\boldsymbol{u})\cdot\boldsymbol{\nabla}^T(\Delta\boldsymbol{u}^*)\\ &= \boldsymbol{\nabla}_s(\Delta\boldsymbol{u}^*) + \boldsymbol{\nabla}(\Delta\boldsymbol{u}^*)\cdot\boldsymbol{\nabla}^T(\boldsymbol{u}^t + \Delta\boldsymbol{u})\end{aligned} \tag{17}$$

Once substituted into the weak form of the equilibrium equation, Eqs. (16) and (17) provide, for the left hand side term of Eq. (1) —strain energy term—,

$$\begin{aligned}\int_{\bar\Gamma}\int_{\Omega(t)} \boldsymbol{E}^* : \mathbf{C} : \boldsymbol{E}\, d\Omega d\bar\Gamma = \int_{\bar\Gamma}\int_{\Omega(t)} &\left(\boldsymbol{\nabla}_s(\Delta\boldsymbol{u}^*) + \boldsymbol{\nabla}(\Delta\boldsymbol{u}^*)\cdot\boldsymbol{\nabla}^T(\boldsymbol{u}^t + \Delta\boldsymbol{u})\right) : \mathbf{C}\\ &: \left(\boldsymbol{\nabla}_s\left(\boldsymbol{u}^t + \Delta\boldsymbol{u}\right) + \frac{1}{2}\left(\boldsymbol{\nabla}(\boldsymbol{u}^t + \Delta\boldsymbol{u})\cdot\boldsymbol{\nabla}^T(\boldsymbol{u}^t + \Delta\boldsymbol{u})\right)\right) d\Omega d\bar\Gamma.\end{aligned} \tag{18}$$

DOI: 10.1002/cnm

The simplest linearization of Eq. (18) consists of keeping in the formulation only constant terms and those linear in $\Delta\boldsymbol{u}$. We thus arrive at a weak form composed by ten terms:

$$
\begin{aligned}
&\int_{\bar{\Gamma}}\int_{\Omega(t)} \boldsymbol{E}^* : \mathbf{C} : \boldsymbol{E} d\Omega d\bar{\Gamma} \\
&\quad = \underbrace{\int_{\bar{\Gamma}}\int_{\Omega(t)} \boldsymbol{\nabla}_s(\Delta\boldsymbol{u}^*) : \mathbf{C} : \boldsymbol{\nabla}_s\boldsymbol{u}^t d\Omega d\bar{\Gamma}}_{T1} + \underbrace{\int_{\bar{\Gamma}}\int_{\Omega(t)} \boldsymbol{\nabla}_s(\Delta\boldsymbol{u}^*) : \mathbf{C} : \boldsymbol{\nabla}_s(\Delta\boldsymbol{u}) d\Omega d\bar{\Gamma}}_{T2} \\
&\quad + \underbrace{\int_{\bar{\Gamma}}\int_{\Omega(t)} \boldsymbol{\nabla}_s(\Delta\boldsymbol{u}^*) : \mathbf{C} : \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{u}^t \cdot \boldsymbol{\nabla}^T\boldsymbol{u}^t d\Omega d\bar{\Gamma}}_{T3} \\
&\quad + \underbrace{\int_{\bar{\Gamma}}\int_{\Omega(t)} \boldsymbol{\nabla}_s(\Delta\boldsymbol{u}^*) : \mathbf{C} : \boldsymbol{\nabla}\boldsymbol{u}^t \cdot \boldsymbol{\nabla}^T(\Delta\boldsymbol{u}) d\Omega d\bar{\Gamma}}_{T4} \\
&\quad + \underbrace{\int_{\bar{\Gamma}}\int_{\Omega(t)} \boldsymbol{\nabla}(\Delta\boldsymbol{u}^*) \cdot \boldsymbol{\nabla}^T\boldsymbol{u}^t : \mathbf{C} : \boldsymbol{\nabla}_s\boldsymbol{u}^t d\Omega d\bar{\Gamma}}_{T5} \\
&\quad + \underbrace{\int_{\bar{\Gamma}}\int_{\Omega(t)} \boldsymbol{\nabla}(\Delta\boldsymbol{u}^*) \cdot \boldsymbol{\nabla}^T\boldsymbol{u}^t : \mathbf{C} : \boldsymbol{\nabla}_s(\Delta\boldsymbol{u}) d\Omega d\bar{\Gamma}}_{T6} \\
&\quad + \underbrace{\int_{\bar{\Gamma}}\int_{\Omega(t)} \boldsymbol{\nabla}(\Delta\boldsymbol{u}^*) \cdot \boldsymbol{\nabla}^T\boldsymbol{u}^t : \mathbf{C} : \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{u}^t \cdot \boldsymbol{\nabla}^T\boldsymbol{u}^t d\Omega d\bar{\Gamma}}_{T7} \\
&\quad + \underbrace{\int_{\bar{\Gamma}}\int_{\Omega(t)} \boldsymbol{\nabla}(\Delta\boldsymbol{u}^*) \cdot \boldsymbol{\nabla}^T\boldsymbol{u}^t : \mathbf{C} : \boldsymbol{\nabla}\boldsymbol{u}^t \cdot \boldsymbol{\nabla}^T(\Delta\boldsymbol{u}) d\Omega d\bar{\Gamma}}_{T8} \\
&\quad + \underbrace{\int_{\bar{\Gamma}}\int_{\Omega(t)} \boldsymbol{\nabla}(\Delta\boldsymbol{u}^*) \cdot \boldsymbol{\nabla}^T(\Delta\boldsymbol{u}) : \mathbf{C} : \boldsymbol{\nabla}_s\boldsymbol{u}^t d\Omega d\bar{\Gamma}}_{T9} \\
&\quad + \underbrace{\int_{\bar{\Gamma}}\int_{\Omega(t)} \boldsymbol{\nabla}(\Delta\boldsymbol{u}^*) \cdot \boldsymbol{\nabla}^T(\Delta\boldsymbol{u}) : \mathbf{C} : \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{u}^t \cdot \boldsymbol{\nabla}^T\boldsymbol{u}^t d\Omega d\bar{\Gamma}}_{T10} .
\end{aligned}
\tag{19}
$$

This renders a very simple scheme that has revealed, however, for judicious choice of the time step Δt, reasonable convergence properties, as will be demonstrated in Section 4.

Remark 1

The original work of P. Ladeveze on the LATIN method [24] combined a space-time separated representation, and thus produces a non-incremental solution of the problem. Generalized to this case, the displacement would be sought in the form $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{s}, t)$. We have preferred, for simplicity of exposition, to keep the formulation as simple as possible, but the explicit linearization proposed in Eq. (19) is by no means the only possible one.

Remark 2

Another possible choice for the before mentioned linearization is the standard forward-Euler scheme. It has been noted, however, that instabilities in the results appear at zones subjected to compression, a typical characteristic of Kirchhoff-Saint Venant models [9]. These are analyzed in Section 4 below. However, no spurious deformation modes have been observed by employing the simple explicit algorithm stated in Eq. (19).

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4. NUMERICAL RESULTS

4.1. Validation: non-linear rod

To validate the explicit PGD approach introduced in Section 2 before, we considered the simple case of a Kirchhoff-Saint Venant beam subjected to a pure traction force F. In this simple case, the model can be simplified to a one-dimensional one, and an analytical solution for the axial displacement u at the bar tip can be obtained as

$$u = \left(\beta - 1 + \frac{1}{3\beta}\right), \tag{20}$$

where

$$\beta = \left(K + \sqrt{K^2 - \frac{1}{27}}\right)^{\frac{1}{3}}, \tag{21}$$

and, in turn, $K = \frac{F}{EA}$, with E the Young's modulus of the material and A the area of the cross-section at the undeformed configuration.

A model was thus constructed by considering a bar of length $L = 400\ mm$, $A = 40 \times 40\ mm^2$, $E = 1.0$ MPa and $F = 320$ N. Load F, however, is considered to be applied at any point along the bar axis. We therefore compute a two-dimensional solution $u = u(x, s)$, where x represents the position along the bar axis and s the point of application of the load.

This simple example served to know the importance of the chosen time step in the overall convergence properties of the proposed method. As can be noticed from Fig. 1 the error for a time step of 10^{-3} is $\mathcal{O}(10^{-4})$. Note that one single PGD approach is used, that is enriched at each time step, not a different PGD approximation within each time step.

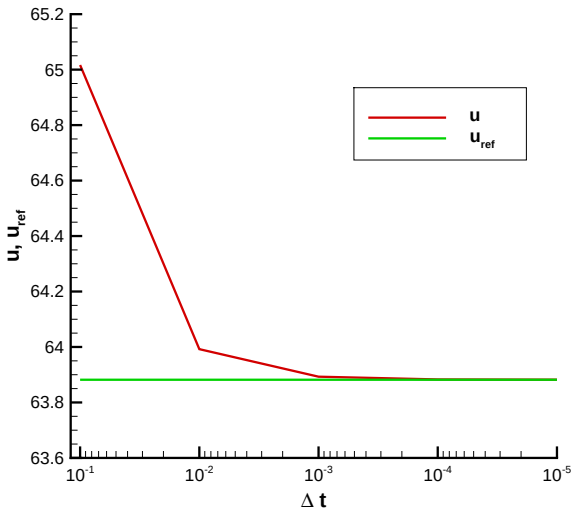

Figure 1. Convergence of the tip displacement towards the reference solution as a function of the chosen time step.

It is important at this point to remark that the real-time strategy here introduced is based upon the computation of a general, high dimensional solution of the problem once for life. This general solution is then evaluated at real time feedback rates, but not re-computed. That is why the time taken in the off-line computation of this general solution is not so important, since it will be done only once.

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4.2. Kirchhoff-Saint Venant beam bending

Further validation of the proposed strategy is obtained if we consider the problem of a beam bending under a transverse load (assumed vertical, for simplicity) applied at any point of its boundary. The problem has no analytical solution, up to our knowledge, and therefore the general, multidimensional solution has been compared to a reference one obtained by standard finite element models (one for each considered load position) with consistent linearization and a Newton-Raphson iterative scheme.

The mesh is composed by tetrahedral elements, with only 9×9 nodes in the $40 \times 40\ mm^2$ cross-section and 21 nodes in the longitudinal direction, $400\ mm$ long. Material parameters were Young's module $E = 1.0$ MPa and Poisson's coefficient $\nu = 0.25$. With such a poor discretization, and employing tetrahedral elements, it is expected that a high error with respect to the exact solution will be obtained. However, it is not the purpose of this paper to obtain an accurate enough solution of the problem (that of course could be obtained by employing a more refined mesh and a finer time stepping).

The finite element model is shown in Fig. 2 and the load is assumed to be applied at any of the points of the upper surface of the beam.

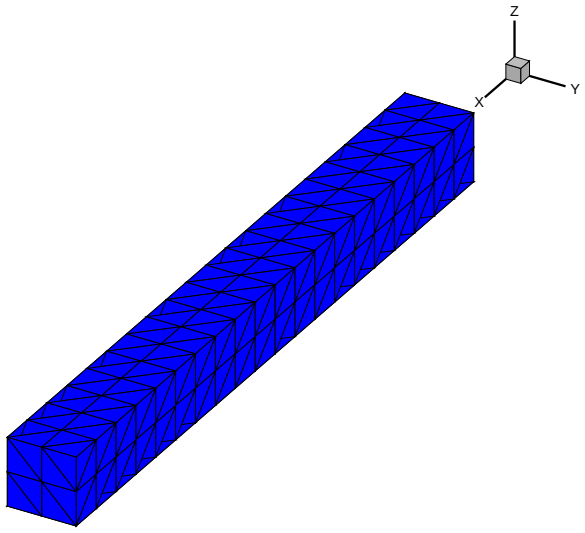

Figure 2. Model for the beam bending problem.

The obtained displacement for a particular location of the load (beam tip in this case, for comparison purposes) is depicted in Fig. 3. Noteworthy, the simple explicit linearization algorithm here proposed does not imply a non-physical gain of volume in the deformed model, which is the case for purely linear elastic models, frequently employed in real-time simulation of surgery [17].

Standard forward-Euler algorithms for the linearization of the weak form of the problem rendered, in our experiments, spurious deformation modes. Although they are intrinsic of the Kirchhoff-Saint Venant constitutive model, it continues to be popular among the virtual surgery community since it constitutes the fastest way to avoid spurious gain in volume typical of linear elastic models, when large deformations are being considered [17]. An example of the result given by a forward-Euler scheme for this same problem is shown in Fig. 4 below. Although no unphysical gain in volume —typical of linear elastic approaches to the problem— is seen, the obtained displacement at beam tip is much higher than the reference one, obtained by finite element methods. This is due to some well-known instabilities of the Kirchhoff-Saint Venant model under compression.

In general, the number n of functions employed in the approximation depends on the desired level of accuracy. Increasing the number of separated functions in the approximation leads, of course, to

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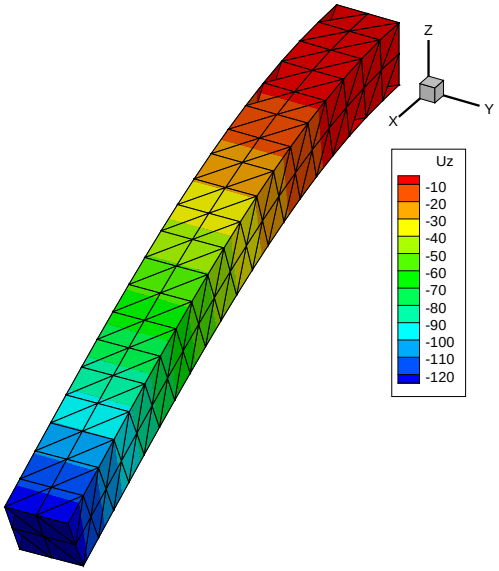

Figure 3. Deformed beam for a particular location of the point load. Note that no unphysical gain in volume is observed.

higher computational costs in the off-line part of the method. But we highlight that this computation is done only once for life, and stored in memory. The on-line part of the simulation is virtually not affected, since only some vector multiplications should be performed in addition, which do not alter the overall efficiency f the proposed method.

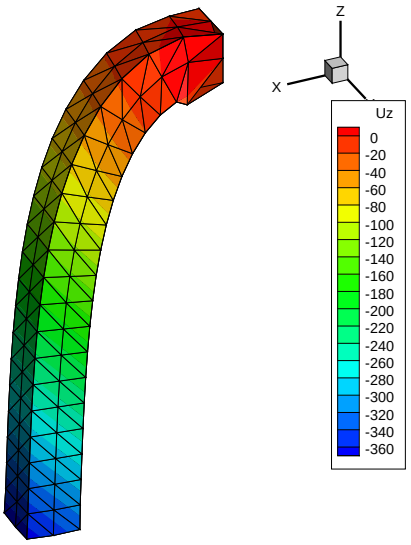

Figure 4. Spurious deformation obtained by employing standard forward-Euler schemes. Note the instabilities near the beam clamping due to compressive stresses.

4.3. Palpation of the liver

The liver is the biggest gland in the human body, after the skin. Liver geometry has been obtained from the SOFA project [3] and post-processed in order to obtain a mesh composed by 8559 nodes

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and 10519 tetrahedra, see Fig. 5. The liver is connected to the diaphragm by the coronary ligament so it seems reasonable to assume it to be constrained at the posterior face by the rest of the organs, while the anterior face is accessible to the surgeon. The inferior vena cava travels along the posterior surface, and the liver is frequently assumed clamped a that location. Although the assumed boundary conditions are not strictly correct from a physiological point of view, our main interest is to show that the model can be solved under real-time constraints with reasonable accuracy.

Although the literature on the mechanical properties of the liver is not very detailed, we have assumed a Young's modulus of 160 kPa, and a Poisson coefficient of 0.48, thus nearly incompressible [17].

The $\bar{\Gamma}$ surface, where the load can be located, has been defined as the whole boundary of the domain, even if in this case, only the frontal part of the organ is usually accessible to the surgeon. This region includes 2009 of the 8559 nodes of the model.

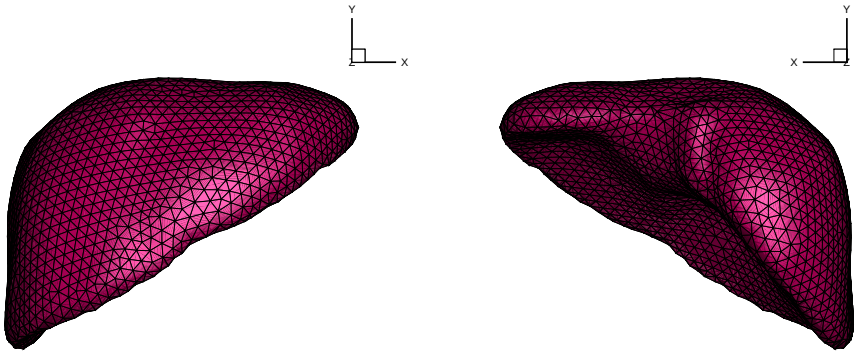

Figure 5. Finite element model for the human liver.

Model's solution was composed by a total of $n = 167$ functional pairs $X_j^k(\boldsymbol{x}) \cdot Y_j^k(\boldsymbol{s})$ (see Eq. (3)). The third component (thus $j = 3$) of the first six modes $X_3^k(\boldsymbol{x})$ is depicted in Fig. 6. The same is done in Fig. 7 for functions $\boldsymbol{Y}$, although in this case they are defined only on the boundary of the domain, i.e., $\bar{\Gamma} = \partial\Omega$.

Noteworthy, both $\boldsymbol{X}$ and $\boldsymbol{Y}$ sets of functions present a structure similar to that generated by Proper Orthogonal Decompositions methods, despite the fact that they are not, in general, optimal. Note how the frequency content of each pair of functions increases as we increase the number of the function, k.

The solution provided by the method agrees well with reference FE solutions obtained employing full-Newton-Raphson iterative schemes (following the same tendency than that shown for the beam bending problem). But, notably, the computed solution can be stored in a so compact form that an implementation of the method is possible on handheld devices such as smartphones and tablets. For instance, for Android-operated devices, an application has been developed (we call it iPGD and is freely downloadable from [10]) that runs the model on a Motorola Xoom tablet running Android 3.0 without problems (only the surface of the model is represented for simplicity, given the limitations of the Android OS). See Fig. 8. The 25 Hz feedback rate necessary for continuous visual perception is achieved without problems.

For more sophisticated requirements, such as those dictated by haptic peripherals, a simple laptop (in our case a MacBook pro running MAC OSX 10.7.4, equipped with 4 Gb RAM and an Intel core i7 processor at 2.66 GHz) is enough to achieve this performance. Feedback rates in the order of kHZ are obtained without problems.

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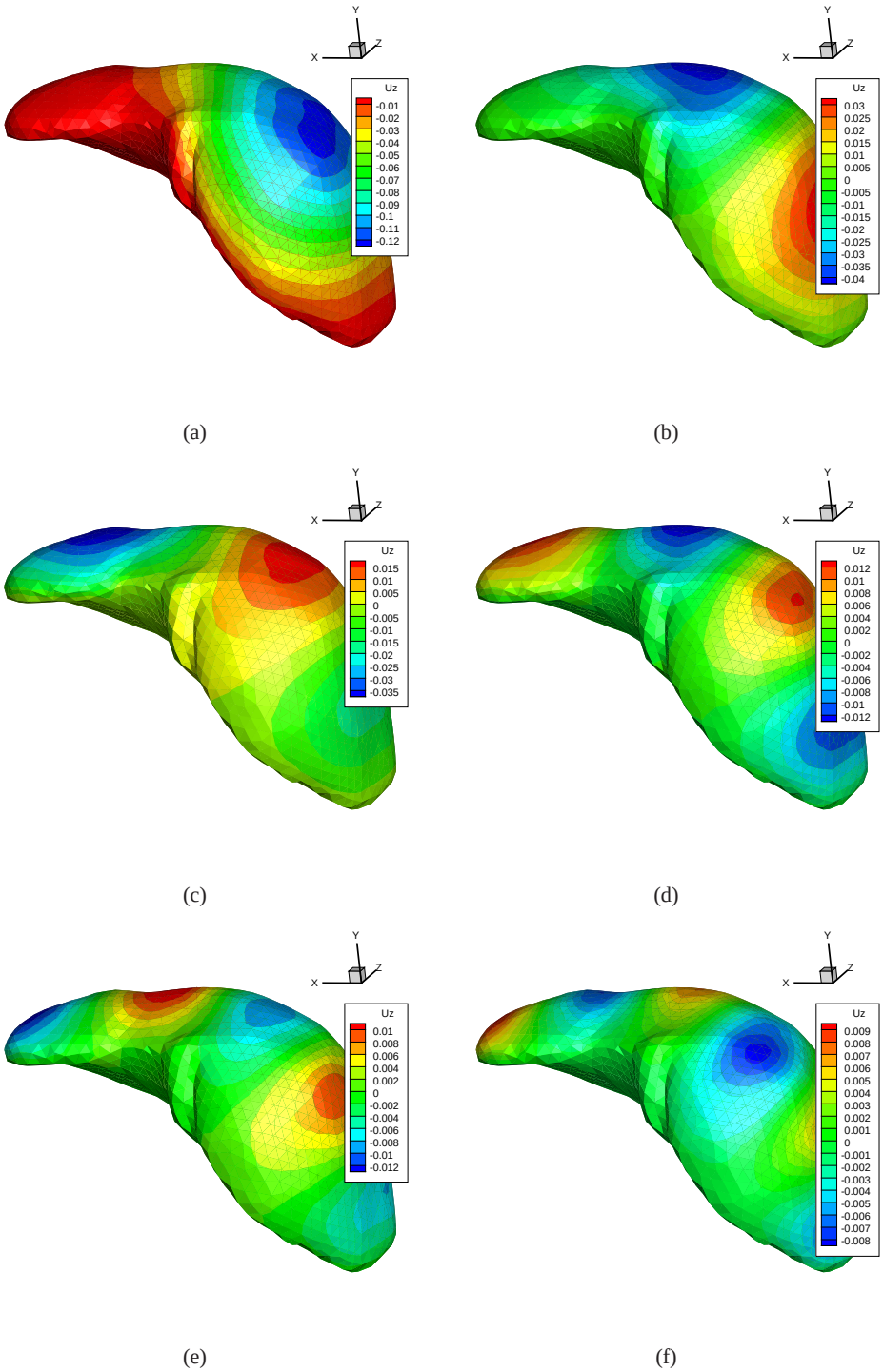

Figure 6. Six first functions $X_3^k(\boldsymbol{x})$, $k = 1, \dots 6$, for the simulation of the liver.

5. CONCLUSIONS

Model order reduction seems to play an important role in real-time simulation of soft biological tissues. In the last times there have been a number of publications on the use of POD techniques

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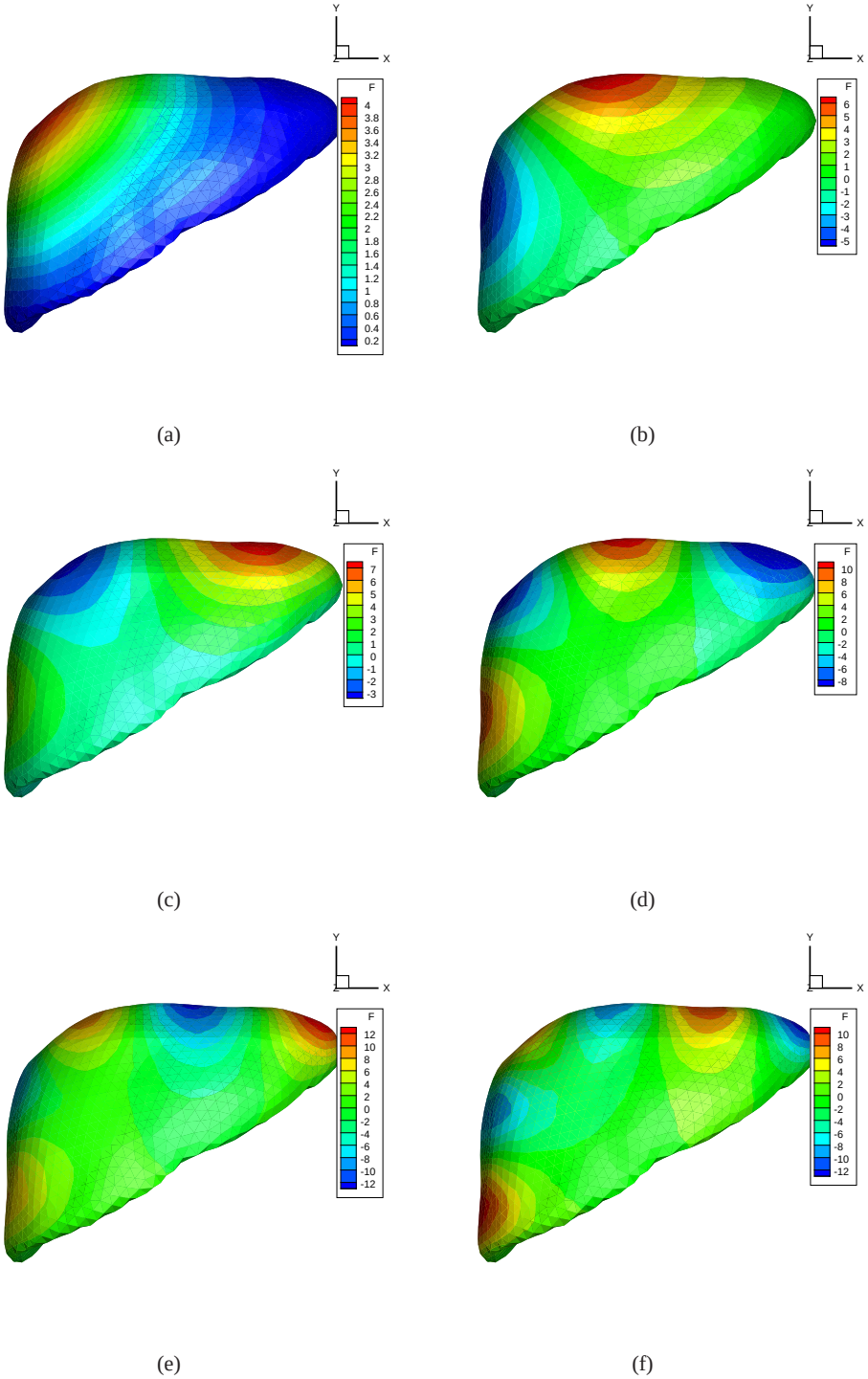

Figure 7. Six first functions $Y_3^k(\boldsymbol{s})$, $k = 1, \dots 6$, for the simulation of the liver. Note that, in this case, functions $\boldsymbol{Y}^k(\boldsymbol{s})$ are defined on the boundary of the liver only.

to this class of problems. However, POD-based approaches seem to over-simplify models and approaches other than [28] do not reproduce properly the non-linearity of soft tissues. In this paper a new approach to the problem has been introduced. It is based on the use of Proper Generalized

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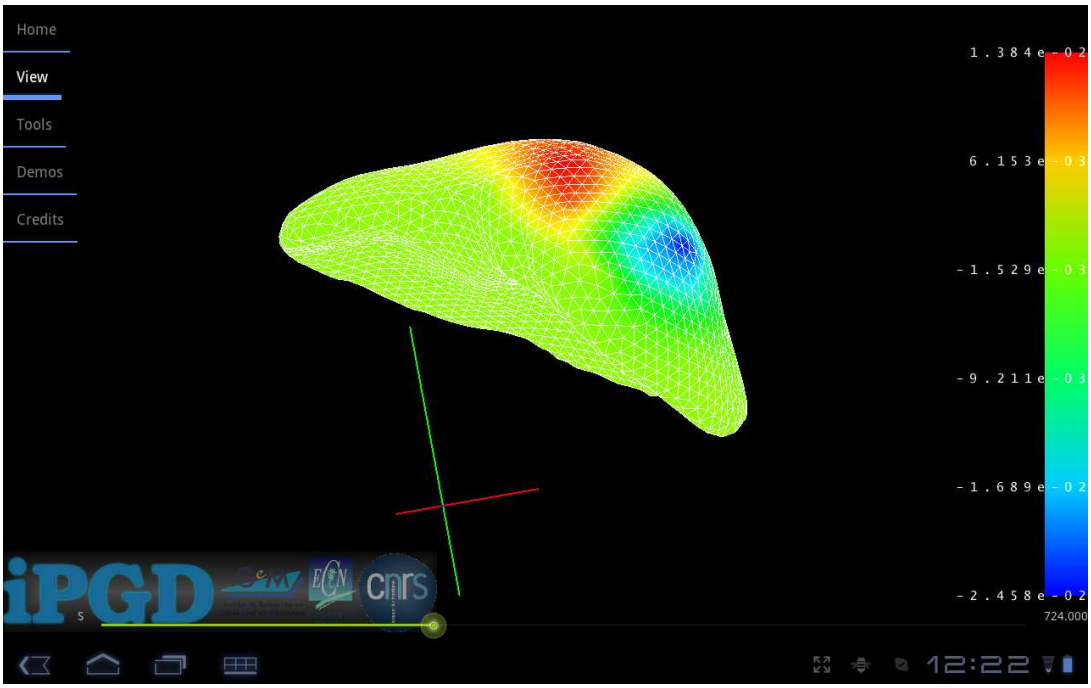

Figure 8. An example of the implementation of the iPGD application for the liver problem.

Decomposition methods. This implies a complete change of paradigm, since PGD (in contrast to POD) do not need for prior computer experiments to generate the snapshots needed to construct the optimal basis functions. The reduced approximation bases, in a separated form, are constructed on the fly.

PGD, on the contrary, operate in a two-stage approach. Firstly, a general meta-model is computed *a priori* once for life. During this intensive computation phase the solution to the high dimensional model is computed as a finite sum of separable functions. This compact solution, although not optimal, in general, provides with a very light format to store the solution in the form of a meta-model that provides the solution to the problem for any parameter value. In this particular problem, parameters are chosen as the position of the contact force between organ and surgical tool (scalpel) and orientation of the load (thus rendering a problem defined in $\mathbb{R}^9$).

This meta-model is then evaluated under real-time restrictions very efficiently (reaching more than 1kHZ in a MacBook pro laptop, for instance). In this paper some benchmark examples have been given to justify the accuracy of the proposed approach. In particular, two different explicit algorithms for time integration of the resulting equations have been proposed. These algorithms have shown to work well, although the development of more robust strategies of linearization of the weak form of the problem, based on the use of Asymptotic Numerical Methods, are now being sought. This is part of our current effort of research.

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