Ph.D. Thesis on Computational Methods in Applied Sciences and Engineering.

Mixed approximations in natural neighbour Lagrange-Galerkin methods. Application to Solid and Fluid Mechanics

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Abstract

State of the art prior to the Thesis

This thesis belongs to the research line devoted to numerical methods in engineering being conducted in the Group of Structural Mechanics and Material Modelling (GEMM) of the Aragón Institute of Engineering Research (I3A), University of Zaragoza. The institute is directed by prof. Doblaré and the particular research line this thesis belongs to by prof. E. Cueto. Both professors acted as advisors of the thesis.

The thesis continues the research line on natural neighbour Galerkin methods opened by the Ph. D. dissertation of prof. E. Cueto in 2001. Natural neighbour Galerkin methods, also known as Natural Element Methods (NEM in what follows) belong to the wide family of meshless methods, like the Element Free Galerkin Methods [3] or the Reproducing Kernel Particle Methods [8] just to cite a few. Meshless methods became popular in the second half of the last decade due to his lack of dependency on a mesh. Although frequently a mesh is necessary to perform numerical integration, its requirements in regularity and conformity are much less demanding that those of the finite element method. Remarkably, the generation of the nodal connectivity is performed in a process transparent to the user, thus alleviating the meshing stage of the construction of a model.

Many meshless methods were constructed by employing different scattered data approximation techniques, like moving least squares or similar techniques. This had, nevertheless, some drawbacks. Among them, the lack of interpolatory character of the approximation, that in turn lead to difficulties in the imposition of essential boundary conditions, or the lack of accuracy in the numerical integration of the weak form of the problem due to the non-polynomial approximation technique used. See [2] for an in-deep study of these problems.

The natural element method arose in the Solid Mechanics community after the Ph. D. of N. Sukumar in Northwestern [10], although some previous references existed [4]. Among the relative advantages of this *new* method one can cite the interpolatory character of the approximation and thus the *exact* imposition —up to linear completeness— of essential boundary conditions in convex domains. Non-convex domains remained an opened issue until the Ph. D. thesis of E. Cueto [5]. By means of the use of an appropriate *neighbourhood* concept, arising from the geometrical concept of α-shapes [7], the authors were able to ensure linear interpolation (bi-linear in three-dimensions) along the essential boundary of the domain.

Other problems, like the most suitable quadrature scheme to be employed to avoid errors in the integration of non-polynomial approximations remained. Although the NEM seemed to be an appealing choice for the simulation in Fluid and Solid Mechanics from a Lagrangian perspective due to the low sensitivity to mesh distortion, other problems remained by that time. For instance, natural neighbour approximation possesses linear completeness at most. Thus, the development of suitable, stable, mixed approximations for incompressible media (i.e., formulations compliant with the LBB or *inf-sup* condition) remained an interesting topic.

Overview of the thesis

The core of the thesis is devoted to the development of a natural neighbour interpolation-based Galerkin method able to be employed in a Lagrangian framework for the simulation of incompressible media.

After a brief state-of-the-art description in Chapter 1 and an overview of meshless methods in Chapter 2, the Natural Element Method is deeply reviewed in Chapter 3. Chapter 4 is

devoted to the problem of finding a suitable quadrature scheme able to be employed with such non-polynomial interpolation. Errors in the numerical quadrature of the weak form of the problem were found to be due to two main causes: (i) the lack of conformity of the integration cells to the shape functions' support and (ii) the rational character of the shape functions. The first aspect received much more attention in the literature by that time (see e.g. [6]), while in the numerical experiments performed, integration errors persisted even if this non-conformity was eliminated.

A stabilized conforming nodal integration scheme due to J. S. Chen and coworkers was employed. This scheme assumes a linearized form of the strain on the Voronoi cells surrounding a node, then performing a boundary integral by employing the divergence theorem. This scheme showed an excellent level of accuracy throughout the numerical test performed.

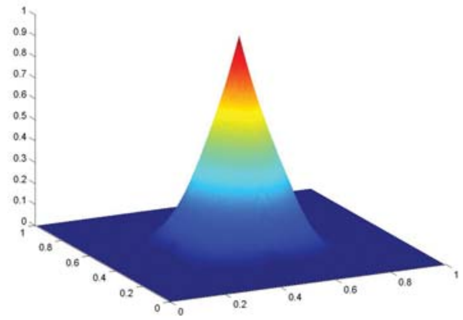

Figure 1: Función de forma de Elementos Naturales.

In Chapter 5 a study of the stability of natural element methods in incompressible media was performed. It was shown that the behaviour of the standard natural element closely resembles that of the bilinear, two-dimensional finite element. In order to improve the reproducing conditions of the approximation, natural neighbour interpolants were enriched by employing Babuška's Partition of Unity paradigm [1], thus acquiring quadratic or even higher consistency. Notably, it was found that a natural element enriched with the monomial $\{1, x \cdot y\}$, and thus acquiring quadratic consistency in displacements (velocities) and constant or linear consistency in pressures seemed to numerically verify the LBB condition. This element closely resembled the well-known Arnold's MINI finite element.

Finally, Chapter 6 was devoted to the development of a Lagrangian approximation to the Navier-Stokes equations. The inertial term of the equations was integrated in time by employing the method of characteristics, since the nodal paths coincide with the characteristic lines of the differential equation. This simple approach gave excellent results in the simulations of some classic problems, like the water column collapse (or broken-dam problem), the simulation of sloshing or the movement of a bubble in a more dense fluid.

Publications arising from the thesis

Parts of this thesis were submitted for publication and some of them are published in leading journals at this time. Chapter 4 was published as "Numerical integration in Natural Neighbour Galerkin methods", D. Gonzalez, E. Cueto, M. A. Martinez and M. Doblare. *International Journal for Numerical Methods in Engineering*, 60(12):2077-2104, 2004.

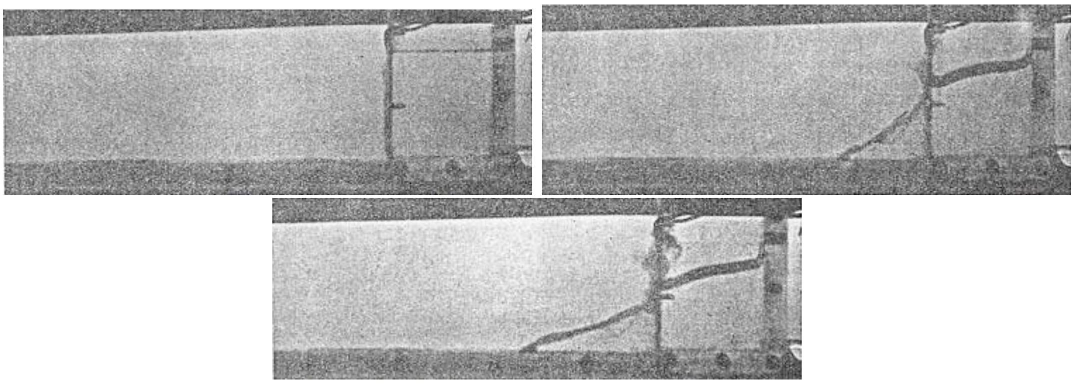

Figure 2: Colapso bidimensional de una columna de agua (tomado de [9]).

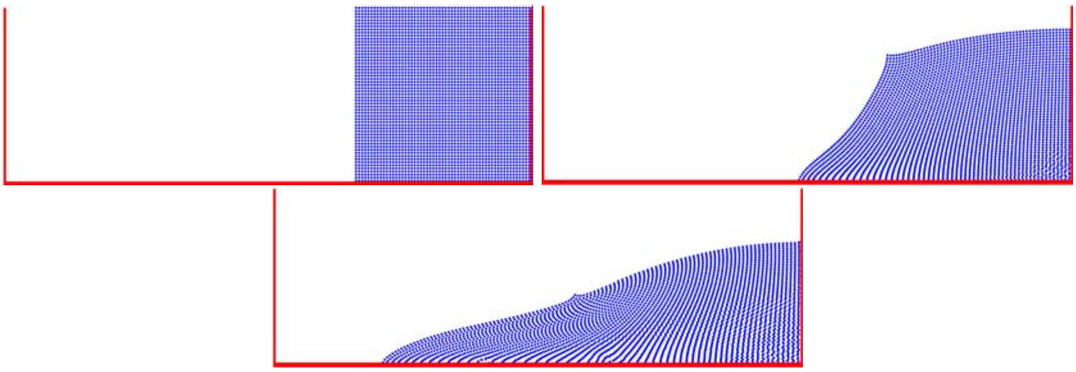

Figure 3: Simulación del colapso bidimensional de una columna de agua.

Chapter 5 was published as "Volumetric locking in Natural Neighbour Galerkin methods", D. González, E. Cueto and M. Doblaré. *International Journal for Numerical Methods in Engineering*, vol. 61(4) pp. 611-632, 2004.

A wide overview of the method including the scheme based on the method of characteristics was published as "The alpha-shape based Natural Element Method in Solid and Fluid Mechanics", D. Gonzalez, I. Alfaro, E. Cueto, M. Doblare and F. Chinesta. Lecture Notes in Computational Science and Engineering. Meshfree Methods for Partial Differential Equations II, Michael Griebel and Marc A. Schweitzer, Ed., Springer-Verlag, pp. 55-70, 2005. Finally, this technique has been also applied recently to the simulation of some forming processes involving important inertial effects, and will soon appear as "Recent advances in the meshless simulation of aluminium extrusion and other related forming processes", by I. Alfaro, D. Gonzalez, D. Bel, E. Cueto, M. Doblare, F. Chinesta in a special number of *Archives of Computational Methods in Engineering*.

A PDF file of the thesis can be downloaded from `http://mmcyte7.cps.unizar.es/ThesisDGonzalez`.

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